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THE METABELIAN p-GROUPS OF MAXIMAL CLASS. II

BY

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ABSTRACT. This paper gives a classification, up to isomorphism, of the metabelian p-groups of maximal class. A recent idea is used to produce a considerable simplification of an earlier classification scheme for these groups.

This paper gives a classification of the metabelian p-groups of maximal class. A recent idea is used to produce a considerable simplification of an earlier classification scheme for these groups.

The usual notation will be employed here. G is a group, $G_2 = [G, G]$ is the commutator subgroup of G, $G_{i+1} = [G_i, G]$ for $i \geqslant 2$. G is of maximal class if and only if $|G| = p^n$, $|G_i : G_{i+1}| = p$ for $1 = 2, \ldots, n-1$. If G is of maximal class then, by definition, G_1 is the largest subgroup of G such that $[G_1, G_2] \leqslant G_4$. It is known that G_1 is a characteristic subgroup of index p in G. The basic facts about these groups can be found in [**1**].

The p-groups of maximal class with G_1 of nilpotence class at most 2 were determined by Leedham-Green in a sequence of three papers [**2**]. His results cover "most" of the groups here for if G is metabelian and (roughly) $n \geqslant 2p$ then G_1 is of class 2. However, the main difficulty with the metabelian groups of maximal class, as far as classification is concerned, is that the class of G_1 can be quite large. When $|G| = p^n$ it can be as large as $(n-1)/2$. This fact led to a difficult proof and complicated statements in [**3**]. This paper gives a way around these problems.

We begin with a description:

THEOREM 1. *Let p be a prime with $p \geqslant 5$. Let G be a metabelian p-group of maximal class and of order p^n where $n \geqslant 5$. Then there is a pair of generators $\{x, y\}$ of G, a basis $\{u_2, \ldots, u_{n-1}\}$ of G_2, an integer k, and a set of integers $\{a(k), \ldots, a(n-1), w, z\}$ with $a(k) \not\equiv 0 \bmod p$ such that $G_1 = \langle y, G_2 \rangle$ and*

(i) $u_2 = [y, x]$, $u_{i+1} = [u_i, x]$ *for* $i \geqslant 2$,

(ii) $[u_2, y] = u_k^{a(k)} \cdots u_{n-1}^{a(n-1)}$,

(iii) $x^p = u_{n-1}^w$, $y^p u_2^{\binom{p}{2}} \cdots u_p^{\binom{p}{p}} = u_{n-1}^z$,

(iv) $u_i^{\binom{p}{1}} \cdots u_{i+p-1}^{\binom{p}{p}} = 1$ *for* $i \geqslant 2$.

The integer k is an invariant of the group and

$$k \geqslant \max\{4, n-p+2\}.$$

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The two kinds of Stirling numbers, S_n^m and $\mathbb{S}_n^m$, will be needed in what follows. They are defined by

$$(x)_n = \sum_{m=0}^{n} S_n^m x^m, \qquad x^n = \sum_{m=0}^{n} \mathbb{S}_n^m (x)_m.$$

By convention $S_n^m = \mathbb{S}_n^m = 0$ for $m > n$. It will also be convenient to define $\sigma(m, n)$ and $\lambda(m, n)$ by

$$\sigma(m, n) = \frac{m!}{n!} S_n^m \quad \text{and} \quad \lambda(m, n) = \frac{m!}{n!} \mathbb{S}_n^m.$$

This brings us to

THEOREM 2. *Let G be defined as in Theorem* 1 *and suppose that* $2k - 3 \leqslant n - 1$. *Then there is an x in G such that* $G = \langle x, G_1 \rangle$,

$$[y, x, y] = u_k^{a(k)} \cdots u_{2k-3}^{a(2k-3)} \mod G_{2k-2}$$

and

$$\sum_{i=k}^{2k-3} \lambda(i - 2, 2k - 5) a(i) = 0.$$

Furthermore, $\langle x, G_2 \rangle$ *is a characteristic subgroup of index p in G.*

We shall utilize Theorem 2 by replacing (ii) of Theorem 1 by an equivalent defining relation. To this end let

$$s(i, j; q, r) = \sigma(j + 1, r + 1)\sigma(i - j - 1, q - r - 1).$$

Next, let

$$u(q, r) = [y, \underbrace{x, \ldots, x}_{q-1-r}, \underbrace{y, \ldots, y}_{r}].$$

Define $U(i, j)$ for $i = 3, \ldots, n - p$ and $j = 1, \ldots, i - 2$ and for $i = n - p + 1, \ldots, n - 1$ and $j = 0, \ldots, i - 2$ by

$$U(i, j) = \prod_{q=i}^{n-1} \prod_{r=j}^{j+q-i} u(q, r)^{s(i,j;q,r)}.$$

The fact that the $U(i, j)$ are well defined for the specified pairs (i, j) is discussed in the paragraphs preceeding Lemma 10.

If $|G| = p^n$ where $n \leqslant p + 1$ then G_2 is of exponent p so the $U(i, j)$ are defined as above for $i = 2, \ldots, n - 1$ and $j = 0, \ldots, i - 2$.

The $U(i, j)$ have the property

$$U(i, j) \equiv u(i, j) \mod G_{i+1}.$$

In addition $\{U(n - p + 1, 0), \ldots, U(n - 1, 0)\}$ is a basis for G_{n-p+1}. Consequently the equation given in (ii) of Theorem 1 is equivalent to one of the form

$$U(3, 1) = \prod_{i=k}^{n-1} U(i, 0)^{b(i)}$$

where $b(k) \neq 0$. If we suppose we have replaced the equation of Theorem 1 by the one above we shall say that G is defined in terms of the parameters $\{b(k),\dots, b(n-1), w, z\}$.

One other preliminary result needs to be stated:

THEOREM 3. *Suppose that G and $\bar{G}$ are isomorphic metabelian p-groups of maximal class. Let θ: $\bar{G} \to G$ be an isomorphism from $\bar{G}$ to G. Define τ from $\bar{G}$ to G by*

$$\bar{x}^{\tau} = \bar{x}^{\theta} g, \qquad \bar{y}^{\tau} = \bar{y}^{\theta} h$$

where g and h are any fixed elements of G_2. Then τ is an isomorphism from $\bar{G}$ to G.

This brings us to our main result:

THEOREM 4. *Suppose that G is a metabelian p-group of maximal class that is defined in terms of the parameters $\{b(k),\dots,b(n-1), w, z\}$. Thus*

$$U(3,1) = U(k,0)^{b(k)} \cdots U(n-1,0)^{b(n-1)}$$

where $b(k) \neq 0$. Suppose also that if $2k-3 \leqslant n-1$ then $G = \langle x, y\rangle$ where x is the element of Theorem 2. Suppose $\bar{G}$, defined in terms of $\{\bar{b}(k),\dots,\bar{b}(n-1), w, z\}$ is isomorphic to G under ϑ where

$$\bar{x}^{\vartheta} = x^{\alpha} y^{\beta}, \qquad \bar{y}^{\vartheta} = y^{\delta},$$

and $\beta = 0$ if $2k-3 \leqslant n-1$. Then

$$\delta b(i) = \alpha^{i-2}\bar{b}(i)$$

for $i = k,\dots,n-1$. Furthermore if $2k-3 \leqslant n-1$ then $b(2k-3) = 0$.

Theorem 4 gives the results that are derived from the commutator structure of the groups. There are two other equations that come out of the power structure:

THEOREM 5. *Let G and $\bar{G}$ be as in Theorem 4. Set $\bar{\psi} = \bar{b}(k)$ if $k = n-p+2$ and $\bar{\psi} = 0$ otherwise. Then*

$$z = \alpha^{n-2}\left[\bar{z} - 2\bar{\psi}\,(\beta/\delta)\right], \qquad w = \alpha^{n-3}\left[\bar{w}\delta - \bar{z}\beta + \bar{\psi}\,(\beta^2/\delta)\right].$$

The equations of Theorems 4 and 5 are necessary and sufficient conditions for isomorphism. In addition, if we take $\bar{G} = G$ and $\bar{b}(i) = b(i)$ in Theorem 4 and use the result of Theorem 3 we obtain the automorphism group of G.

Some of the results of [3] are used here. Theorem 3 is Lemma 1.1 of [3]. Theorem 5 is part of Theorem 2 of [3]. The main purpose of this paper is to eliminate the complicated analysis that runs from p. 101 to p. 118 in [3].

1. A function must be defined. Set $F_\alpha(0,0) = 1$ and $F_\alpha(0,n) = 0$ for $n \geqslant 1$. For $n \geqslant 1$ let

$$F_\alpha(m,n) = \sum_x \binom{\alpha}{x_1} \cdots \binom{\alpha}{x_m}$$

where the summation is on those $x = (x_1,\dots,x_m)$ with integral $x_i \geqslant 1$ for $i = 1,\dots,m$ and $x_1 + \cdots + x_m = n$.

LEMMA 1. *Let*

$$e(n, i, j) = \binom{\alpha}{i+1} \sum_{t=0}^{j-n} \binom{\beta}{t} F_\delta(n+1, j+1-t)$$

and

$$f(m, M, N) = \sum_{a,b} \frac{m!}{a!b!(m-a-b)!} F_\alpha(m-a, M) F_\beta(m-b, N)$$

where the summation is on all those integers a, b with $a \geqslant 0$, $b \geqslant 0$ *and* $a + b \leqslant m$. *Set*

$$g(m, n, r, s) = \sum_{i=0}^{r} \sum_{j=n}^{s} e(n, i, j) f(m, r-i, s-j).$$

Let x and y be elements of a metabelian group and set $u = [y, x]$. *Set* $x_1 = x^\alpha y^\beta$, $y_1 = y^\delta$, *and* $u_1 = [y_1, x_1]$. *Then*

$$[u_1, \underbrace{x_1, \ldots, x_1}_{m}, \underbrace{y_1, \ldots, y_1}_{n}] = \prod_{\substack{r \geqslant 0 \\ r+s \geqslant m+n}} \prod_{s \geqslant n} [u, \underbrace{x, \ldots, x}_{r}, \underbrace{y, \ldots, y}_{s}]^{g(m,n,r,s)}.$$

PROOF. See [**4**, Lemma 2.1] for the details.

LEMMA 2. *Let* $g(m, n, r, s)$ *be defined as in Lemma* 1. *Then*

$$g(m, n, r, n) = \delta^{n+1} F_\alpha(m+1, r+1)$$

and

$$g(m, n, m-1, n+1) = m\alpha^m \delta^{n+1} \beta.$$

This follows directly from the definitions.

LEMMA 3. *Suppose G is defined as in Theorem* 1, $x_1 = x^\alpha y^\beta$, $y_1 = y^\delta$ *and* $u_1 = [y, x]$. *Then*

$$[u_1, \underbrace{x_1, \ldots, x_1}_{m}] \equiv \prod_{s=0}^{1} \prod_{r \geqslant m-s} [u_{r+2}, \underbrace{y, \ldots, y}_{s}]^{g(m,0,r,s)} \mod G_{2k-2}.$$

PROOF. If G is of maximal class then, in Lemma 1,

$$[u, \underbrace{x, \ldots, x}_{r}] = u_{r+2}.$$

Furthermore if $[u_2, y]$ is defined by (ii) in Theorem 1 then, for $i \geqslant 2$

$$[u_i, y] \equiv u_{k+i-2}^{a(k)} \mod G_{k+i-1}.$$

Thus if we set aside those commutators corresponding to $s \geqslant 2$ in the conclusion of Lemma 1 we obtain Lemma 3.

LEMMA 4. *Suppose that* $\bar{G}$ *and* G *are isomorphic under* ϑ *where* $\bar{x}^\vartheta = x^\alpha y^\beta$ *and* $\bar{y}^\vartheta = y^\delta$. *Suppose also that*

$$[\bar{u}_2, \bar{u}_1] = \prod_{m=k}^{n-1} (\bar{u}_m)^{\bar{a}(m)}.$$

Then

$$[\bar{u}_2, \bar{u}_1]^{\vartheta} \equiv \prod_{r=k}^{2k-3} u_r^{h(r)} \mod G_{2k-2}$$

where

$$h(r) = \delta \sum_{m=k}^{r} \bar{a}(m) F_\alpha(m-1, r-1) + \varepsilon(r)(k-2)\alpha^{k-2}\delta\bar{a}(k)a(k)\beta$$

with $\varepsilon(r) = 0$ *for* $r \leqslant 2k-4$ *and* $\varepsilon(2k-3) = 1$.

PROOF. This follows from

$$\prod_{m \geqslant k} \left(\bar{u}_m^{\vartheta}\right)^{\bar{a}(m)} = \prod_{m \geqslant k} [u_1, \underbrace{x_1, \ldots, x_1}_{m-2}]^{\bar{a}(m)},$$

Lemma 3, and Lemma 2. The first part of $h(r)$ comes from that part of the product in Lemma 3 corresponding to $s = 0$. The $\varepsilon(r)$ term comes from the main part of $[u_{k-1}, y]^{\tau}$ where $\tau = \bar{a}(k)g(k-2, 0, k-3, 1)$.

LEMMA 5. *Suppose the hypotheses of Lemma* 4 *hold. Then*

$$\left[\bar{u}_2^{\vartheta}, \bar{u}_1^{\vartheta}\right] \equiv \prod_{r=k}^{2k-3} u_r^{i(r)} \mod G_{2k-2}$$

where for $n = k, \ldots, 2k-3$

$$i(r) = \delta^2 \sum_{m=k}^{r} a(m) F_\alpha(1, r+1-m).$$

PROOF. This follows from Lemma 1. Take $m = 0$ and $n = 1$. Several Stirling number identities are needed at this point.

LEMMA 6. *Let* $F_\alpha(m, n)$ *be defined as above. Then*

$$F_\alpha(m, n) = \sum_{t=m}^{n} \lambda(m, t)\sigma(t, n)\alpha^t.$$

In addition

$$\sum_{m=a}^{M-b} \sigma(a, m)\sigma(b, M-m) = \sigma(a+b, M),$$

$$\sum_{l=j+c}^{N} \sigma(j, l-c)\lambda(l, N) = \lambda(c, N-j),$$

$$\sum_{m=a}^{b} \sigma(a, m)\lambda(m, b) = \delta(a, b)$$

where $\delta(a, b) = 1$ *if* $a = b$ *and* $\delta(a, b) = 0$ *if* $a \neq b$.

PROOF. See §3 of [**4**] for the proofs.

LEMMA 7. *Let*

$$h_0(r) = \delta \sum_{m=k}^{r} \bar{a}(m) F_\alpha(m-1, r-1)$$

and

$$\bar{a}(m) = \sum_{\nu=k}^{m} \bar{P}(\nu)\sigma(\nu-2, m-2).$$

Let

$$\varphi(r, \nu, \alpha) = \sum_{t=1}^{r+1-\nu} \lambda(1, t)\sigma(t+\nu-2, r-1)\alpha^t.$$

Then

$$h_0(r) = \delta \sum_{\nu=k}^{r} \alpha^{\nu-2}\bar{P}(\nu)\varphi(r, \nu, \alpha).$$

PROOF. This follows from the identities of Lemma 6.

Incidentally, if $\bar{a}(k),\ldots,\bar{a}(m)$ are considered as given, then $\bar{P}(k),\ldots,\bar{P}(m)$ are functions of $\bar{a}(k),\ldots,\bar{a}(m)$. Since $\sigma(m-2, m-2) = 1$ the number $\bar{P}(m)$ is well defined.

LEMMA 8. *Let*

$$i(r) = \delta^2 \sum_{m=k}^{r} a(m) F_\alpha(1, r+1-m)$$

and

$$a(m) = \sum_{\nu=k}^{m} P(\nu)\sigma(\nu-2, m-2).$$

Let $\varphi(r, \nu, \alpha)$ be defined as in Lemma 7. Then

$$i(r) = \delta^2 \sum_{\nu=k}^{r} P(\nu)\varphi(r, \nu, \alpha).$$

PROOF. Apply Lemma 6.

LEMMA 9. *Suppose that $\bar{G}$ is isomorphic to G under ϑ, $2k-3 \leqslant n-1$, $\bar{a}(m)$ is defined as in Lemma 7, and $a(m)$ is defined as in Lemma 8. Then*

$$\delta P(r) = \alpha^{r-2}\bar{P}(r) \quad \textit{for } r = k,\ldots,2k-4$$

and

$$\delta P(2k-3) = \alpha^{2k-5}\bar{P}(2k-3) + (k-2)\alpha^{k-3}a(k)\bar{a}(k)\beta.$$

PROOF. This follows from Lemmas 4 through 8.

PROOF OF THEOREM 2. First, if $2k-3 \leqslant n-1$ then $k-2 \not\equiv 0 \bmod p$. For if $k-2 \equiv 0 \bmod p$ then, since $k \geqslant 4$, $k = 2 + t \cdot p$ where t is a positive integer. Since $2k-3 \leqslant n-1$ it follows that $n \geqslant 2p+2$. But if $n \geqslant 2p+2$ then, since $k \geqslant n-p+2$, we have $2k-3 \geqslant n$.

Thus the coefficient of β in the last equation of Lemma 9 is not zero. Consequently if $\bar{G}$ is considered as given we can find an isomorphic image of G with $P(2k-3)=0$.

Next, by the definition of the $a(m)$ given in Lemma 2 and by the last identity of Lemma 6

$$\sum_{m=k}^{2k-3} \lambda(m-2, 2k-5)a(m) = P(2k-3).$$

Thus there is an x such that $G = \langle x, G_1 \rangle$ and such that the above sum is zero.

Finally $\langle x, G_2 \rangle$ is a characteristic subgroup of G. To see this apply Lemma 9 with $\bar{x} = x$, $x^{\vartheta} = x^{\alpha}y^{\beta}$ $\bar{a}(m) = a(m)$, and $\bar{P}(2k-3) = P(2k-3) = 0$. Then the last equation of Lemma 9 is

$$0 = (k-2)\alpha^{k-3}a(k)\bar{a}(k)\beta.$$

Thus $\beta = 0$, so $x^{\vartheta} = x^{\alpha}$ and $\langle x, G_2 \rangle$ is a characteristic subgroup of G. It is of index p because G is generated by two elements.

We need to show that the $U(i, j)$ are well defined. We had

$$U(i, j) = \prod_{q=i}^{n-1} \prod_{r=j}^{j+q-i} u(q, r)^{s(i,j,q,r)}$$

for $i = 3, \ldots, n-p$ and $j = 1, \ldots, i-2$ and for $i = n-p+1$ and $j = 0, \ldots, i-2$.

To begin $s(i, j, q, r) = \sigma(j+1, r+1)\sigma(i-j-1, q-r-1)$. Then, from the definitions

$$|\sigma(a, b)| = \sum_{(x_1, \ldots, x_a)} \frac{1}{x_1 \cdots x_a}$$

where the summation is on those integral $(x_1, \ldots, x_a)$ with $x_i \geqslant 1$ and $x_1 + \cdots + x_a = b$. Thus $\sigma(a, b)$ is a well-defined residue modulo p if and only if $b - a \leqslant p-2$. That is $s(i, j; q, r)$ is a well-defined residue modulo p if and only if $r - j \leqslant p-2$ and $q - r - (i-j) \leqslant p-2$.

Next if $r - j \geqslant p-1$ or $q - r - (i-j) \geqslant p-1$ and $u(q, r)$ appears in the definition of $U(i, j)$ then $u(q, r) = 1$. Consider first those cases where $r \geqslant 1$. Then

$$u(q, r) = [u_2, \underbrace{x, \ldots, x}_{q-2-r}, \underbrace{y, \ldots, y}_{r}] = [u_k^{a(k)} \cdots u_{n-1}^{a(n-1)}, \underbrace{x, \ldots, x}_{q-2-r}, \underbrace{y, \ldots, y}_{r-1}].$$

Now if $r - j \geqslant p-1$ then $r \geqslant p-1$ so $k + r - 1 \geqslant k + p - 2 \geqslant n$. If $q - r - (i-j) \geqslant p-1$ then $q - r - 2 \geqslant p-1$ and $k + q - 2 - r \geqslant k + p - 1 \geqslant n$. In either event $u(q, r) = 1$. Consider next the case when $r = 0$. Then, by the definition of $U(i, j)$, $j = 0$. But if $j = 0$ then $i \geqslant n - p + 1$, and $q - r - (i-j) = q - i \leqslant n - 1 - (n-p+1) = p-2$. Thus the two inequalities mentioned in the last line of the previous paragraph hold.

In the sum, if $u(q, r)$ appears in the definition of $U(i, j)$ then the corresponding exponent is a well-defined residue modulo p.

Finally, by Theorem 1, G_{n-p+1} is of exponent p and $G_k \leqslant G_{n-p+2}$. Consequently if $r \geqslant 1$ or if $r = 0$ and $q \geqslant n-p+1$ then $u(q, r) \in G_{n-p+1}$. That is, all the $u(q, r)$ appearing in the definition of the $U(i, j)$ are elements of order p.

LEMMA 10. *Suppose $\bar{G}$ is isomorphic to G under ϑ where $\bar{x}^{\vartheta} = x^{\alpha}$ and $\bar{y}^{\vartheta} = y^{\delta}$. Suppose that $2k - 3 \leqslant n - 1$. Let $U(i, j)$ be defined as in the introduction. Then*

$$\bar{U}(i, j)^{\vartheta} = U(i, j)^{\varepsilon(i,j)}$$

where $\varepsilon(i, j) = \alpha^{i-j-1}\delta^{j+1}$.

PROOF. This is given in Lemmas 3.2 through 3.4 in [4].

LEMMA 11. *Suppose $\bar{G}$ is isomorphic to G under ϑ where $\bar{x}^{\vartheta} = x^{\alpha}y^{\beta}$ and $\bar{y}^{\vartheta} = y^{\delta}$. Suppose that $2k - 3 \geqslant n$. As above, let $\varepsilon(i, j) = \alpha^{i-j-1}\delta^{j+1}$. Then*

$$\bar{U}(3,1)^{\vartheta} = U(3,1)^{\varepsilon(3,1)}$$

and, for $i = k, \ldots, n-1$

$$\bar{U}(i,0)^{\vartheta} = U(i,0)^{\varepsilon(i,0)}.$$

The proof is similar to the one of Lemma 10, but one uses the fact that $[u_i, y, y] = 1$ for all $i \geqslant 2$.

To prove the last equation, for example, start from the definition of the $U(i, j)$ to get

$$\bar{U}(i,0) = \prod_{q=i}^{n-1} \prod_{r=0}^{q-i} [\bar{u}_2, \underbrace{\bar{x},\ldots,\bar{x}}_{q-2-r}, \underbrace{\bar{y},\ldots,\bar{y}}_{r}]^{s(i,0,q,r)}.$$

Since $[u_i, y, y] = 1$ those commutators above where $r \geqslant 2$ can be dropped. Furthermore for $r = 1$ one has

$$[\bar{u}_2, \underbrace{\bar{x},\ldots,\bar{x}}_{q-3}, \bar{y}] = [\bar{u}_{q-1}, \bar{y}] \in \bar{G}_{q-1+k-2}.$$

Since $q + k - 3 \geqslant 2k - 3 \geqslant n$ the last group is $\langle 1 \rangle$. Thus we have

$$\bar{U}(i,0) = \prod_{q=i}^{n-1} [\bar{u}_2, \underbrace{\bar{x},\ldots,\bar{x}}_{q-2}]^{s(i,0,q,0)}.$$

To continue, apply ϑ to $\bar{U}(i,0)$ and apply Lemma 1. This produces a multiple product of commutators. Next, use the considerations that appeared in the previous paragraph to set aside those commutators in this multiple product that have one or more "y". One then has

$$\bar{U}(i,0)^{\vartheta} = \prod_{q \geqslant i} [u_2, \underbrace{x,\ldots,x}_{q-2}]^{m(i,q)}$$

where

$$m(i,q) = \sum_{t=i}^{q} g(t-2,0,q-2,0)s(i,0,t,0).$$

Now, by definition, $s(i,0,t,0) = \sigma(i-1, t-1)$. So, by Lemma 2,

$$m(i,q) = \delta \sum_{t=i}^{q} F_{\alpha}(t-1, q-1)\sigma(i-1, t-1).$$

Then, by Lemma 6,

$$m(i,q) = \delta\alpha^{i-1}\sigma(i-1,q-1) = \varepsilon(i,0)s(i,0,q,0).$$

This proves that $\bar{U}(i,0)^{\vartheta} = U(i,0)^{\varepsilon(i,0)}$.

The $\delta b(i) = \alpha^{i-2}\bar{b}(0)$ equation of Theorem 4 now follows from Lemmas 10 and 11.

We shall now prove that $b(2k-3)=0$. By part (ii) of Theorem 1 and the definition of $U(3,1)$,

$$U(3,1) \equiv \prod_{q \geqslant k} u_q^{A(q)} \mod G_{2k-2}$$

where

$$A(q) = \sum_{i=k}^{q} a(i)\sigma(1,q+1-i).$$

Next, if $i > k$ then

$$U(i,0) \equiv \prod_{q\geqslant i} \big[u, \underbrace{x,\dots,x}_{q-2}\big]^{s(i,0,q,0)} \mod G_{2k-2}.$$

To see this, rearrange the double product that defines $U(i,0)$ as

$$U(i,0) = \prod_{r\geqslant 0}\ \prod_{q\geqslant i+r} \big[u, \underbrace{x,\dots,x}_{q-2-r}, \underbrace{y,\dots,y}_{r}\big]^{s(i,j,q,r)}.$$

Consider those commutators here where $r \geqslant 1$. We have

$$\big[u, y, \underbrace{x,\dots,x}_{q-2-r}, \underbrace{y,\dots,y}_{r-1}\big] \in G_{k+q-3}.$$

Since $q \geqslant i+r \geqslant i+1 \geqslant k+1$ we have $G_{k+q-3} \leqslant G_{2k-2}$. This proves the first assertion of this paragraph.

Consequently,

$$\prod_{i\geqslant k} (U(i,0))^{b(i)} \equiv \prod_{q\geqslant k} u_q^{B(q)} \mod G_{2k-2}$$

where

$$B(q) = \sum_{i=k}^{q} b(i)\sigma(i-1,q-1).$$

Next, multiply the equation $A(q) = B(q)$ by $\lambda(q-1,m-1)$ and sum from $q=k$ to $q=m$ to obtain, with the aid of Lemma 6,

$$b(m) = \sum_{i=k}^{m} \lambda(i-2,m-2)a(i).$$

To finish take $m = 2k - 3$. Then, by the choice of the $a(i)$,

$$b(2k-3) = \sum_{i=k}^{2k-3} \lambda(i-2, 2k-5)a(i) = 0.$$

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